Lecture 10- Schauder || Calderon-Zygmund

- ∘ Statement
- ∘ Examples and counterexamples

C^α Schauder 1930s
L^p Calderon-Zygmund 1950s
<u>Interior</u> $\triangle u = f(x)$
+ Positive
$f \in C^\alpha(B_1) \;\; \Rightarrow D^2 u \in C^\alpha\left(B_{1/2}\right)$
$f \in \underset{1<p<\infty}{L^p}(B_1) \;\Rightarrow D^2 u \in L^p\left(B_{1/2}\right)$

$f \in BMO \;\Rightarrow D^2 u \in BMO$
$f \in H^1 \;\Rightarrow D^2 u \in H^1$

- Negative
$f \in C^0 \;\not\Rightarrow D^2 f \in C^0$
$f \in L^{1/\infty} \not\Rightarrow D^2 f \in L^{1/\infty}$

Boundary $\begin{cases} \triangle u = 0 \;\; x_n > 0 \\ u(x,0) = g(x) \;\; x_n = 0 \end{cases}$
+ Positive
$g \in C^0, C^\alpha \;\; \Rightarrow \;\; u \in C^0, C^\alpha$
$g \in C^{1,\alpha} \;\; \Rightarrow \;\; u \in C^{1,\alpha}$
$g \in C^{2,\alpha} \;\; \Rightarrow \;\; u \in C^{2,\alpha}$

$g \in L^1, L^p, L^\infty \;\; \Rightarrow \;\; u \in L^1, L^p, L^\infty$
$g \in \underset{1<p<\infty}{W^{1,p}} \;\; \Rightarrow \;\; u \in W^{1,p}$
$g \in \underset{1<p<\infty}{W^{2,p}} \;\; \Rightarrow \;\; u \in W^{2,p}$
Switch p to BMO or H^1 OK.
- Negative
$g \in C^1 \;\; \not\Rightarrow \;\; u \in C^1$
$g \in C^2 \;\; \not\Rightarrow \;\; u \in C^2$

$g \in W^{1,1}, W^{1,\infty} \;\; \not\Rightarrow \;\; u \in W^{1,1}, W^{1,\infty}$
$g \in W^{2,1}, W^{2,\infty} \;\; \not\Rightarrow \;\; u \in W^{2,1}, W^{2,\infty}$

Examples/Counterexamples
Interior
Eg 0.

$$u = h(x)(\ln r)^{1/3} \quad \text{with } h(x) = \operatorname{Im} z^k$$
$$\left|D^k u(0)\right| = \infty$$
$$\triangle u = O\left(r^{k-2}\frac{1}{(\ln r)^{2/3}}\right) \in C^0, C^{k-2}.$$

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More expensive ones.

$$u(x) = \Gamma * f(x) = \int_{\mathbb{R}^n} \begin{matrix} \dfrac{c_n}{|x-y|^{n-2}} & \\ \ln|x-y| & n=2 \\ |x-y| & n=1 \end{matrix} \; f(y)\,dy \quad \text{for } f \in C_0^{\alpha}(\mathbb{R}^n).$$

Exercise: Try at least for $f \in C_0^{\infty}(\mathbb{R}^n)$, see [Gilbarg-Trudinger, p. 55 Lem4.2].

$$\begin{aligned}
D_i u(x) &= \int_{\mathbb{R}^n} \Gamma_i(x-y) f(y)\,dy, \\
D_{ij} u(x) &= \lim_{\substack{R\to\infty \\ \varepsilon\to 0}} \int_{B_R(x)\setminus B_\varepsilon(x)} \Gamma_{ij}(x-y)\left[f(y) - f(x)\right] dy \\
&= \lim_{\substack{R\to\infty \\ \varepsilon\to 0}} \int_{B_R\setminus B_\varepsilon} \Gamma_{ij}(y)\left[f(x-y) - f(x)\right] dy = \lim_{\substack{R\to\infty \\ \varepsilon\to 0}} \int_{B_R\setminus B_\varepsilon} \Gamma_{ij}(y) f(x-y)\,dy \\
&= \int_{\mathbb{R}^n} C_n \frac{y_i y_j}{|y|^{n+2}}\left[f(x-y) - f(x)\right] dy \neq \int_{\mathbb{R}^n} C_n \frac{y_i y_j}{|y|^{n+2}} f(x-y)\,dy, \\
D_{ii} u(x) &= \lim_{\substack{R\to\infty \\ \varepsilon\to 0}} \int_{B_R(x)\setminus B_\varepsilon(x)} \Gamma_{ii}(x-y)\left[f(y) - f(x)\right] dy + \frac{1}{n} f(x) \\
&= \lim_{\substack{R\to\infty \\ \varepsilon\to 0}} \int_{B_R\setminus B_\varepsilon} \Gamma_{ii}(y)\left[f(x-y) - f(x)\right] dy + \frac{1}{n} f(x) \\
&= \int_{\mathbb{R}^n} C_n \frac{y_i^2 - \frac{1}{n}|y|^2}{|y|^{n+2}}\left[f(x-y) - f(x)\right] dy + \frac{1}{n} f(x) \neq \int_{\mathbb{R}^n} \Gamma_{ii}(x-y) f(y)\,dy.
\end{aligned}$$

Eg 1. $f(x) = \chi_\Omega \in L^\infty(R^2)$, where $\Omega = \{x_1 x_2 \geq 0\} \cap B_1$.

figure

$$\begin{aligned}
u_{12}(0) &= C_2 \int\int_\Omega \frac{y_1 y_2}{|y|^4} f(-y)\,dy \\
&\sim \int\int \frac{\frac{1}{2}\sin 2\theta}{r^2} r\,dr\,d\theta \sim \int \frac{1}{r}\,dr = \infty.
\end{aligned}$$

RMK. One can smooth out f to get smooth counterexamples for $W^{2,\infty}$ estimates.

Eg 2. $f(x) = \frac{1}{|\ln|x||^{1/2}} \chi_{B_{1/2}} \in C^0$. figure?

$$\begin{aligned}
u_{12}(0) &== C_2 \int\int_{B_{1/2}} \frac{y_1 y_2}{|y|^4} \frac{1}{|\ln|y||^{1/2}}\,dy \\
&\sim \int \frac{1}{r} \frac{1}{|\ln r|^{1/2}}\,dr = \infty.
\end{aligned}$$

Eg 3. $f(x)=\frac{1}{\epsilon^n}\varphi\left(\frac{x}{\varepsilon}\right)\in L^1$, $\varphi=\varphi(|x|)\in C_0^\infty$, $\varphi\geq 0$, and $\int\varphi=1$. Say still on $\mathbb{R}^2$

$$\begin{aligned}
u&=\Gamma*f=\Gamma(x)\quad\text{for }\ |x|\geq\varepsilon\\
D_{12}u&=C_2\frac{x_1x_2}{|x|^4}\quad\text{for }\ |x|\geq\varepsilon\\
C\|f\|_{L^1}&\ngtr\|D_{12}u\|_{L^1(B_1)}\geq\int_{B_1\setminus B_\varepsilon}\frac{x_1x_2}{|x|^4}\sim\int_{B_1\setminus B_\varepsilon}\frac{1}{|x|^2}=|\ln\varepsilon|\to\infty
\end{aligned}$$

But $\quad\|f\|_{L^1}=1$.

Boundary situation.

Eg 0'. $u_x=g'\in L^\infty$, but $u_y\notin L^\infty$

$$u=\operatorname{Im}z\log z=x\theta+y\ln r$$

$$u(x,0)=x\theta=\begin{cases}0 & x\geq 0\\ \pi x & x<0\end{cases}$$

$$\begin{aligned}
u_x(x,0)&=\theta\in L^\infty\\
u_y&=\partial_y(x\theta+y\ln r)\quad\text{little tedious, instead}\\
u_y&=\partial_y\operatorname{Im}z\log z=\operatorname{Im}\partial_y(z\log z)\\
&=\operatorname{Im}\frac{1}{i}\left(\bar\partial-\partial\right)(z\log z)=\operatorname{Im}i\partial(z\log z)\quad\text{here }\begin{cases}\bar\partial=\partial_{\bar z}=\frac{1}{2}(\partial_x+i\partial_y)\\ \partial=\partial_z=\frac{1}{2}(\partial_x-i\partial_y)\end{cases}\\
&=\operatorname{Re}\partial(z\log z)=\operatorname{Re}(\log z+1)\\
&=\ln r+1\notin L^\infty.
\end{aligned}$$

$u_x=g'\in C^0$, but $u_y\notin L^\infty$

$$u=\operatorname{Im}z(\log z)^{1/3}=r\left(\ln^2 r+\theta^2\right)^{1/3}\sin\left(\frac{\theta}{3\ln r}+\theta\right)$$

$$u(x,0)=\left\{\begin{array}{ll}0 & x\geq 0\\ |x|\left(\ln^2|x|+\pi^2\right)^{1/3}\sin\left(\frac{\pi}{3\ln|x|}+\pi\right) & x<0\end{array}\right\}\in C^0,\quad\text{of course near 0!}$$

$$\begin{aligned}
u_y&=\operatorname{Re}\partial\left(z\log^{\frac{1}{3}}z\right)=\operatorname{Re}\left[\log^{\frac{1}{3}}z+\frac{1}{3}(\log z)^{-2/3}\right]\\
&=\underbrace{\left(\ln^2 r+\theta^2\right)^{1/3}\cos\left(\frac{\theta}{3\ln r}\right)}_{\notin L^\infty}+\frac{1}{3}\underbrace{\left(\ln^2 r+\theta^2\right)^{-2/3}\cos\left(\frac{-2\theta}{3\ln r}\right)}_{\in C^0}.
\end{aligned}$$

Eg 0" $u_{xx}(x,0)=g''\in L^\infty$, but $u_{xy}\notin L^\infty$.

$$\begin{aligned}
u&=\operatorname{Im}z^2(\log z)=2xy\ln r+\left(x^2-y^2\right)\theta\\
u(x,0)&=\left\{\begin{array}{ll}0 & x\geq 0\\ \pi x^2 & x<0\end{array}\right\}\in C^{1,1}
\end{aligned}$$

$$u_{xx}(x,0) \in L^\infty$$

$$\begin{aligned} u_{xy} &= \partial_{xy} \operatorname{Im} z^2 (\log z) = \operatorname{Im} \frac{1}{i} \left(\bar\partial^2 - \partial^2\right) z^2 (\log z) \\ &= \operatorname{Re} \partial^2 \left[z^2 (\log z)\right] = \operatorname{Re}(\log z + 1) = \ln r + 1 \notin L^\infty \\ u_{yy} &= -2\theta \in L^\infty. \end{aligned}$$

More expensive ones.

Poisson formula

$$u(x) = c_n \int_{R^{n-1}} \frac{x_n}{\left(|\xi|^2 + x_n^2\right)^{n/2}} g(x' - \xi)\, d\xi$$

$$D_{x'} u(x) = u(x) = c_n \int_{R^{n-1}} \frac{x_n}{\left(|\xi|^2 + x_n^2\right)^{n/2}} D_{x'} g(x' - \xi)\, d\xi$$

$$D_{x'x'} u(x) = u(x) = c_n \int_{R^{n-1}} \frac{x_n}{\left(|\xi|^2 + x_n^2\right)^{n/2}} D_{x'x'} g(x' - \xi)\, d\xi$$

$D_{x_n} u(x)$ direct computation messy. Instead $\xi = x_n \bar\xi$

$$u(x) = c_n \int_{R^{n-1}} \frac{1}{\left(\left|\bar\xi\right|^2 + 1\right)^{n/2}} g\left(x' - x_n \bar\xi\right) d\bar\xi$$

$$\begin{aligned} D_{x_n} u(x) &= c_n \int_{R^{n-1}} \frac{-\bar\xi \cdot D_{x'} g\left(x' - x_n \bar\xi\right)}{\left(\left|\bar\xi\right|^2 + 1\right)^{n/2}} d\bar\xi \\ &= c_n \int_{R^{n-1}} \frac{-\xi}{\left(|\xi|^2 + x_n^2\right)^{n/2}} \cdot D_{x'} g(x' - \xi)\, d\xi \end{aligned}$$

$$\begin{aligned} D_{x'x_n} u(x) &= c_n \int_{R^{n-1}} \frac{-\xi}{\left(|\xi|^2 + x_n^2\right)^{n/2}} \cdot D_{x'x'} g(x' - \xi)\, d\xi \\ &= c_n \frac{-x'}{\left(|x'|^2 + x_n^2\right)^{n/2}} * D_{x'x'} g \end{aligned}$$

Now harmonic function in R^n

$$\frac{\xi}{\left(|\xi|^2 + x_n^2\right)^{n/2}} \overset{*}{=} c_n \int_{R^{n-1}} \frac{x_n}{\left(|\xi - x'|^2 + x_n^2\right)^{n/2}} \frac{x'}{|x'|^n} dx' \quad = P_{x_n} * \frac{x'}{|x'|^n}$$

Exercise: Observe the integral make sense, because $P_{x_n} \in L^2\left(\mathbb{R}^{n-1}\right) \cap P_{x_n} \in L^{10.1}\left(\mathbb{R}^{n-1}\right)$ matching the singur kernel $\frac{x'}{|x'|^n}$ on $\mathbb{R}^{n-1}$. Justify *.

Step 1.

$$\frac{\xi}{\left(|\xi|^2 + (x_n + \varepsilon)^2\right)^{n/2}} = c_n \int_{R^{n-1}} \frac{x_n}{\left(|\xi - x'|^2 + x_n^2\right)^{n/2}} \frac{x'}{\left(|x'| + \varepsilon^2\right)^n} dx' \quad .$$

Step 2.

$$\underline{\lim_{\varepsilon\to0+}} c_n \int_{R^{n-1}} \frac{x_n}{\left(|\xi-x'|^2+x_n^2\right)^{n/2}} \frac{x'}{\left(|x'|+\varepsilon^2\right)^n} dx' = \lim_{\varepsilon\to0+} c_n \int_{\underline{R^{n-1}\backslash B_\varepsilon}} \frac{x_n}{\left(|\xi-x'|^2+x_n^2\right)^{n/2}} \frac{x'}{|x'|^n} dx' .$$

See [Stein-Weiss, p. 236 Th'm 4.17, p. 218 Th'm Lem1.2, p. 13 Th'm1.25].

$$u_{x_n} = P_{x_n} * \frac{-x'}{|x'|^n} * D_{x'} g$$

$$u_{x'x_n} = \underset{\text{nice kernel}}{P_{x_n}} * \underset{\text{singular kernel}}{\frac{-x'}{|x'|^n}} * D_{x'x'} g$$

$\frac{x'}{|x'|^n}$ on $\mathbb{R}^{n-1}$ satisfies

· homogeneous degree $n-1$

· $\int_{\partial B_1} \frac{x'}{|x'|^n} = 0$.

This implies $P_{x_n} * \frac{-x'}{|x'|^n} *$ preserves C^α $0<\alpha<1$ and L^p $1<p<\infty$ norm.

It does not preserve C^0, C^1, L^1, L^∞ norm.

- eg1' $g' \in L^\infty$, but $u_y \notin L^\infty$ - eg1" $g'' \in L^\infty$, but $u_{xy} \notin L^\infty$
- eg2' $g' \in C^0$, but $u_y \notin L^\infty$ - eg2" $g'' \in C^0$, but $u_{xy} \notin L^\infty$
- eg3' $g' \in L^1$, but $u_y \notin L^1$ - eg3" $g'' \in L^1$, but $u_{xy} \notin L^1$

Eg1' $\begin{cases} \triangle u = 0 \text{ for } y>0 \\ u(x,0) = g(x) = |x| \in W^{1,\infty} \end{cases}$

$$u(x,y) = C_2 \int_{R^1} \frac{y}{\xi^2+y^2} g(x-\xi)\, d\xi$$

$$u_y(x,y) = C_2 \int_{R^1} \frac{-\xi}{\xi^2+y^2} g'(x-\xi)\, d\xi$$

Now

$$u_y(0,0) = C_2 \int_{R^1} \frac{-1}{\xi} g'(-\xi)\, d\xi = \infty.$$

In fact

$$u = \frac{-2}{\pi}\left(x\theta + y\ln r - \frac{\pi}{2}x\right), \quad \text{see eg 0'}.$$

Eg 2' $g' = \frac{1}{\ln|x|} \in C^0$, yes near 0, or say C^0 $g' = \frac{1}{|\ln|x||^e}$, or $\frac{1}{\ln|x|} + e^{\frac{-1}{|x|}}$ not from holomorphic functions, but $u_y(0,0) = \infty$.

Eg 3' $g' = \frac{1}{\ln|x|} \in L^1$, yes near 0, but $u_y(x,0) \notin L^1$.

Replace g' by g'', and u_y by u_{xy}, we get eg1"2"3".

RMK. Freezing coefficients, general version for

$$\sum a_{ij}(x) D_{ij}u + b_i(x) D_i u + cu = f(x)$$

C^α :

$$a_{ij} \in C^\alpha,\ b_i \in C^\alpha,\ c \in C^\alpha,\ f \in C^\alpha \implies D^2 u \in C^\alpha.$$

L^p :

$$a_{ij} \in C^0 \text{ (or } VMO),\ b_i \in C^0 \text{ (or } L^\infty,\ L^q),\ c \in C^0 \text{ (or } L^\infty,\ L^q),\ f \in L^p \implies D^2 u \in L^p.$$